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DEVELOPING UNDERGRADUATE MATHEMATICS STUDENTS' CONCEPTUAL KNOWLEDGE ABOUT THE SAMPLING DISTRIBUTION OF THE SAMPLE MEAN AND CENTRAL LIMIT THEOREM

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ABSTRACT

This study highlights some practical steps lecturers could utilize to develop undergraduate mathematics students' conceptual knowledge about the sampling distribution of the sample mean and Central Limit Theorem. An intact class of fifty (50) first-year undergraduate mathematics students were selected from a university in Ghana. Through guided inquiry, the students individually, and in their respective groups, completed specific tasks assigned to them by their lecturer. The tasks included, computing the sample means, probabilities, variances, and drawing the sampling distribution of the sample means. The students answered questions from their lecturer to conceptually understand the sampling distribution of the sample mean and how that engages them to understand the Central Limit Theorem. The results indicated that the use of the guided inquiry for instruction greatly enhances students' conceptual knowledge about the sampling distribution of the sample means and the Central Limit Theorem. An implication of this study is that lecturers should allow students to develop mathematical concepts from simple case scenarios, and guide them gradually to transition seamlessly into understanding complex scenarios. Lecturers, therefore, should be encouraged to use technology and simulations to explain statistical concepts to their students.

INTRODUCTION:

The sampling distribution of the sample mean and Central Limit Theorem are very important statistical concepts (Rubin et al., 1990; Shaughnessy, 1992; Tversky & Kahneman, 1971). Yet, they are the least understood, because of the pedagogical methods teachers use to explain them. Conceptual knowledge is the knowledge of concepts that allows students need to comprehend mathematical concepts, operations, and relations (e.g., Byrnes & Wasik, 1991; Siegler, & Alibali, 2001; Kilpatrick et al., 2001; Canobi, 2009; Rittle-Johnson et al., 2001). This knowledge, which could appear in several domains in mathematics learning, should be stated plainly and clearly (Goldin Meadow et al., 1993). This type of knowledge encompasses not only what is known, but also how concepts can be known (e.g., deeply and with rich connections) (Star,2005). Some teachers agree that teaching students to understand mathematical concepts is very paramount in their professional practice. These teachers, through this practice, motivate students to build a solid and sound conceptual knowledge in mathematics.

In the mathematics classroom, teachers must teach concepts first. Conceptual understanding is an important goal in learning mathematics because it is required to make sense of a phenomenon. Understanding entails enabling students to construct, interpret and explain the meaning (Anderson et al., 2001). It involves understanding the principles that govern a domain and the relationships between units of knowledge in a domain (Rittle Johnson et al., 2001). Conceptual understanding enables students to understand the relationship between mathematical ideas and transfer their knowledge into new situations by applying it to new contexts. Emphasis on conceptual understanding now requires teachers to use a teaching style that encourages students to construct their meaningful mathematical concepts, through an inquiry-based model (Boaler, 2008). One of the benefits of conceptual understanding is that students are less likely to forget concepts than procedures. Once the conceptual understanding is developed, it enhances the development of procedural knowledge (Rittle-Johnson et al., 2001). Conceptual understanding ultimately prepares students with adequate procedural knowledge in their mathematical problem-solving skills. When it is combined with procedural knowledge, the combination is much more powerful than either one alone (Wong & Evans, 2007).

LITERATURE REVIEW:

Baroody et al., (2007) opine that conceptual knowledge should be defined as 'knowledge about facts, and principles' (p. 107), and should not emphasize how that knowledge is connected. Support for this assertion comes from research on conceptual change that shows that (1) novices' conceptual knowledge is often not fully constructed and needs to be integrated throughout learning and (2) experts' conceptual knowledge which is fully constructed continues to grow and become better organized (diSessa et al., 2004; Schneider & Stern, 2009).

In developing conceptual understanding, teachers should provide suitable working environments and adhere to practices that would encourage students to work in groups (Vosniadou, 2001). teachers act as a co-ordinators by providing guidance and support in mathematics learning, alongside the development of skills that allow students to work together. The ability to work together in the mathematics classroom is a skill that needs to be taught (Hunter, 2010). Once this is acquired, students can help each other and utilise mathematical reasoning when explaining their ideas to others.

As the importance of the sampling distribution of the means and Central Limit Theorem becomes more and more apparent to students and novice researchers, mathematics teachers still continue to find appropriate and effective ways of introducing the concepts to students. The reliance on abstract concepts and methods have given way to more innovative methods. Computer technology, now allows statistical concepts to be easily demonstrated. A proliferation of computer activities on the sampling distribution of the means and Central Limit Theorem the Central Limit Theorem, is enough to attest to this (e.g., Finzer & Erickson, 1998; Kader, 1990; Kreiger & Pinter-Lucke, 1992; Stirling, 2002). Often, teachers resort to guided inquiry, which involves careful planning, close supervision, authentic assessment and targeted intervention through the inquiry process that gradually leads students toward independent learning (Crede, 2008).

Its ultimate goal is to develop independent thinking-minded persons who can expand their knowledge and expertise through skilful use of a variety of information inside and outside of the school (Kuncel, 2008). Guided inquiry enables students to explore concepts by themselves. This could be achieved when teachers adopt a guided inquiry method for teaching, in combination with an active study habit. This approach will motivate students and build their interest in a lesson. It focuses on students' attention and initiates problem-solving. If guided inquiry and active study habit for teaching are utilized, students will be self-reliant and competent in their class participation.

The study was guided by the following research questions: (1) What practical steps could teachers use to develop mathematics students' conceptual knowledge about the sampling distribution of the sample means? (2) How does this knowledge lead these students to understand the Central Limit Theorem? (2) How do simulations of the sampling distribution of the sample means engage students to understand the Central Limit Theorem?

METHOD:

Participants and Setting of the study:

The study adopted a guided inquiry learning method for students to complete a series of tasks assigned to them by their lecturer. It was conducted at a university in Ghana in the 2019/2020 academic year, using an intact class of fifty (50) students. The students, with a mean age of nineteen (19) years, two months, were first-year undergraduate mathematics students, who had barely spent a semester in the university. A diagnostic set of questions given to the students indicated that they possessed basic knowledge in the elementary theory of probability and statistics.

Tasks and Instructions:

Ten (10) groups, each comprising five (5) students, were formed. Initially, the lecturer met all the students and assigned the groups with five (5) tasks and instructions. For the first, second, third and fourth tasks, the lecturer asked a volunteer from each group to toss the die once, twice, thrice and four times, respectively, while the other group members looked on and observed the outcomes. As a group, they computed the probability for each sample mean and drew the sampling distribution of the means of each of the tosses. For the fifth and sixth tasks, the lecturer asked the students to generate 3 and 6 random numbers between 1 and 6 using Microsoft Excel. For each set of numbers, each group of students computed the sample mean for the 10,000 simulations, and drew the sampling distributions of the means.

In all these tasks, the lecturer adopted guided-inquiry instruction as a form of investigation to support the students, by providing them with adequate resources, such as graph sheets, pencils, calculators, and computers. The lecturer facilitated the learning process, but also sought and learnt more about the students and their thinking pattern. The lecturer emphasized evaluating the development of information-processing skills and conceptual understanding. The following conversations took place between the lecturer and students:

Lecturer: Class, nominate a volunteer from your group to toss the die.
Group 1: We nominate Araba
Lecturer: Araba, toss the die. What are the possible outcomes of your toss?
Araba: The outcomes we observed are 1, 2, 3, 4, 5, 6.
Lecturer: Did the other groups obtain the same outcomes when the die is tossed by your volunteers?
Groups: All responded in the affirmative.
Lecturer: Do you all agree that each of the six outcomes occur once?
Groups: Yes, we do.
Let a volunteer from your group calculate the probability of each outcome. Group 3, what are your results?
Group 3: Kwesi gave their results as 1/6, 1/6, 1/6, 1/6, 1/6, 1/6.
Lecturer: Kwesi, how did you arrive at these probabilities?
Kwesi: We've learnt in class that Probability of an event = Number of times the event occurs/total number of outcomes. So, we applied the concept we learnt in class.
Lecturer: Each group should complete the table with the headings: Random variable (x), Frequency and probability(P(x)), and draw the distribution of *x* against *frequency* on a graph sheet.
I'm coming round to inspect your results. If I see what you've done, I'll assist you to complete the table and draw the distribution (see Table 1 and Fig 1.).
Lecturer: Group 5, would you nominate a volunteer to describe the shape of the distribution?
Kwabena: The graph of the distribution is flat, with constant frequency.
Lecturer: what is the name of this distribution?
Kwabena: It's a uniform distribution.
Lecturer: Good

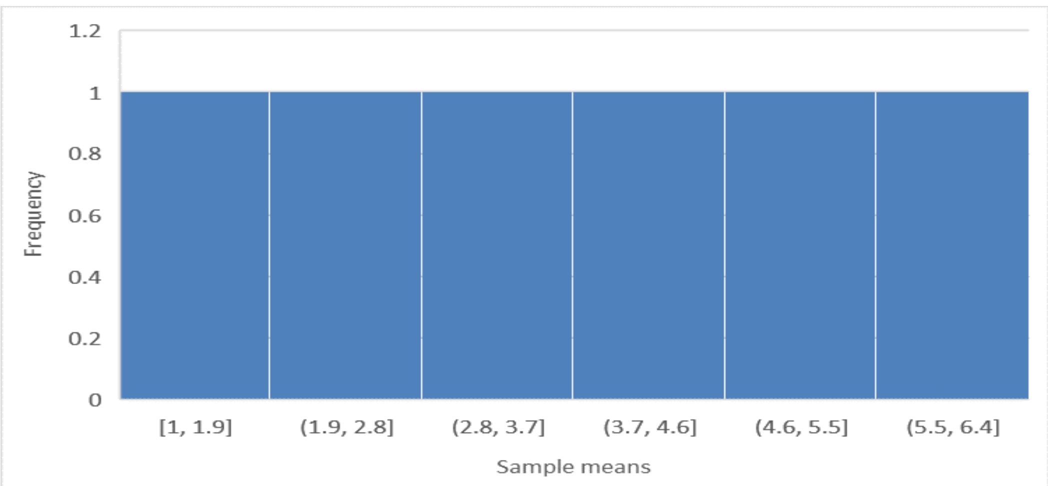
Table 1

Random variable, frequency and probability of each outcome when a die is tossed once

x	$frequency$	$p(x)$
1	1	1/6
2	1	1/6
3	1	1/6
4	1	1/6
5	1	1/6
6	1	1/6

Fig1

Sample distribution of the sample means for a toss of a six-sided die



Lecturer: Each group should compute the mean of the random variable, x.
Group 5: Yaw gave their answer as (1+2+3+4+5+6)/6=7/2=3.5
Lecturer: Do the other groups share the same opinion?
Group 2: Yes, we do.
Lecturer: I want all of you to find the mathematical expectation and variance using the formulae $\mu_x = \sum xp(x)\ and\ \sigma^2{}_x = \sum(x-\mu)^2 p(x)$, respectively.
Lecturer: Group 7, what is your answer for the expectation?
Efua: Our answer is 7/2=3.5.
Lecturer: Group 6, what about the variance?
Kofi: Our answer is 35/12.
Lecturer: You've all done very well. Let's compute the expectation and variance together:

The mean of the random variable x:

$$\mu_x = \sum xp(x) = 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right) = \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6} = \frac{21}{6} = \frac{7}{2} = 3.5$$

The variance of the random variable x:

$$\sigma^2{}_x = \sum(x-\mu)^2 p(x) = (1-3.5)^2\frac{1}{6} + (2-3.5)^2\frac{1}{6} + (3-3.5)^2\frac{1}{6} + (4-3.5)^2\frac{1}{6} + (5-3.5)^2\frac{1}{6} + (6-3.5)^2\frac{1}{6} = \frac{6.25}{6} + \frac{2.25}{6} + \frac{0.25}{6} + \frac{0.25}{6} + \frac{2.25}{6} + \frac{6.25}{6} = \frac{35}{12}$$

Lecturer: All the groups, nominate a volunteer to toss the die twice.
Mansa: Some possible outcomes are (1,1), (1, 2), (1,3), (1,4), (1,5), (1,6), and so on.
All the groups should find the total possible outcomes of the two tosses. Group 3, are you ready to give your answer?
Yaw: Our total possible outcomes is 36.
Lecturer: Great, how did you come by that?
Yaw: Well, we listed all the possible pairs of outcomes and found it to be 36.
Lecturer: Good. Group 4, Could you have done it differently?
Esi: We used the counting principle to arrive at 36.
Lecturer: Please, explain your procedure.
Esi: We know that each of the 6 possible outcomes on the first toss is a possible outcome, and so is each of the 6 possible outcomes on the second toss of the die.
Altogether, we obtain 6 x 6 = 36 outcomes.
Lecturer: All groups should compute the sample means of the 36 observations. For instance, the sample mean of the random variable (3,2) is (3+2)/2=5/2=2.5.
Amina: For Group 6, we've completed ours. Could we show it to the entire class?
That's good. Let's all complete the table together. All groups should complete the table of the sample mean, frequency of the sample mean, and its associated probability. I'm coming round to inspect your work.
Group 8: Members started grumbling.
Lecturer: What's the matter?
Mansa: We're unsure of the probabilities we've computed. Please Sir, check and see if we're on the right track.

You're not wholly right, you're wrong in some few places. I presume all of you were not meticulous in arriving at the frequencies. If you had gotten the frequencies of the sample mean right; you would have obtained the correct probabilities. I'm happy other groups are still figuring out the proper answers. Please, we'll complete the table together by applying the concept of permutation. For example, when a die is tossed twice, the outcomes that give a sum of 2, 3, and 4 are 11, 21, 31 & 22, respectively. The outcome 11 can be obtained in one way, the outcome 21 can be obtained in two ways, the outcomes 31 and 22 can be obtained in two ways and one way respectively. Table 2 indicates the sample mean, numerator sum, permutation, and frequency. Note that 11 can be obtained in 2! /2! ways, 21 can be obtained in 2! Ways, while 31 and 22 can be obtained in 2! ways and 2! /2! ways respectively. Table 2 and 3 indicate the random variables for two tosses of a die and their sample means, and their sample means, frequency of the sample means and its associated probability, respectively.

Table 2
Random variables for two tosses of a die and their sample means

Sample Mean	Numerator Sum	Permutation	Frequency	Sample Mean	Numerator Sum	Permutation	Frequency
2/2	11	2! /2!	1	12/2	66	2! /2!	1
3/2	21	2!	2	11/2	65	2!	2
4/2	31	2!	2	10/2	64	2!	2
	22	2! /2!	1		55	2! /2!	1
Total			3				3
5/2	32	2!	2	9/2	63	2!	2
	14	2!	2		54	2!	2
Total			4				4
6/2	51	2!	2	8/2	62	2!	2
	33	2! /2!	1		53	2!	2
	24	2!	2		44	2! /2!	1
Total			5				5
7/2	61	2!	2				
	52	2!	2				
	43	2!	2				
Total			6				

Table 3
Sample mean, frequency of the sample mean, and its associated probability

$\bar{x}$	Frequency	P ($\bar{x}$)
1	1	1/36
3/2	2	1/18
2	3	1/12
5/2	4	1/9
3	5	5/36

7/2	6	1/6
4	5	5/36
9/2	4	1/9
5	3	1/12
11/2	2	1/18
6	1	1/36

Lecturer: All the groups should go ahead and compute the mean and variance of the sample mean, $\bar{x}$, using the formulae $\mu_{\bar{x}} = \sum \bar{x} p(\bar{x})$ and $\sigma^2{}_{\bar{x}} = \sum(\bar{x} - \mu)^2 p(\bar{x})$ respectively.

Group 9: Have you computed your values?

Adjoa: Yes, we have. Sir, please come and look at them.

Lecturer: You're correct, well done. All the groups should observe if there is any relationship between the mean and the variance when the die is tossed once and twice. Group 8, do you have something to say?

Kweku: We observe that the sample means in the two instances are the same.

Lecturer: A good observation. What about the relationship between the two variances?

Kweku: We're yet to identify any relationship.

Lecturer: The other groups are struggling as well, so, let me guide all of you to identify the relationship. Group 4, divide the variance when the die is tossed once by 2. What is the result?

Kwansema: Our result is 35/12/2=35/24.

Lecturer: What then is your conclusion?

Kwansema: The variance when the die is tossed twice is equal to the variance when it is tossed once divided by 2.
Good, all the groups should go ahead and draw the distribution of the sample means. Group 5, what do you notice about the shape of the distribution?

Kojo: We notice that it is bell-shaped and symmetrical about the mean of 3.5.

Lecturer: As a whole class, let's compute the values together. The computation for the mean and variance is indicated below. The graph of the distribution is indicated in Fig. 2.

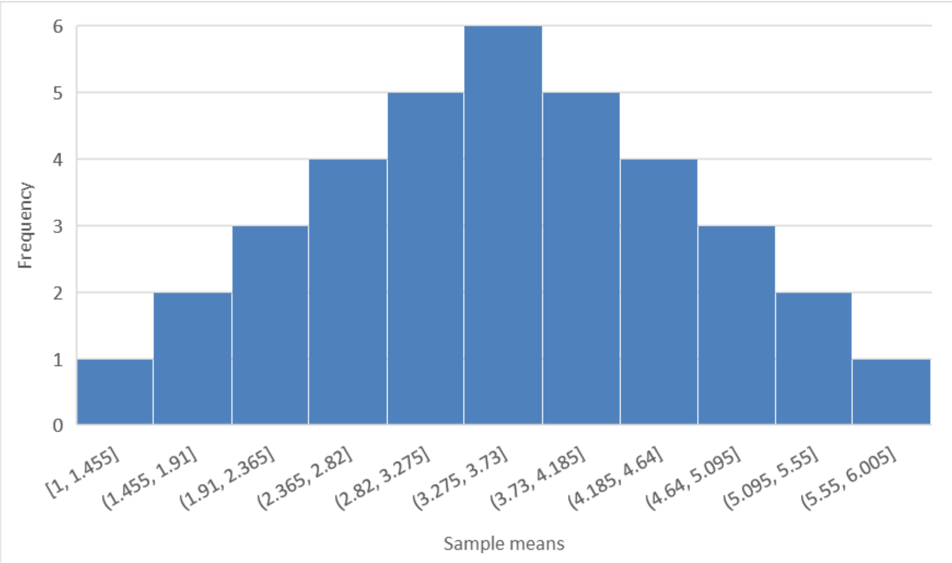
The mean of the random variable $\bar{x}$:

$$\mu_{\bar{x}} = \sum \bar{x} p(\bar{x}) = 1\left(\frac{1}{36}\right) + \frac{3}{2}\left(\frac{1}{18}\right) + 2\left(\frac{1}{12}\right) + \frac{5}{2}\left(\frac{1}{9}\right) + 3\left(\frac{5}{36}\right) + \frac{7}{2}\left(\frac{1}{6}\right) + 4\left(\frac{5}{36}\right) + \frac{9}{2}\left(\frac{1}{9}\right) + 5\left(\frac{1}{12}\right) + \frac{11}{2}\left(\frac{1}{18}\right) + 6\left(\frac{1}{36}\right) = \frac{1}{36} + \frac{3}{36} + \frac{6}{36} + \frac{10}{36} + \frac{15}{36} + \frac{21}{36} + \frac{20}{36} + \frac{18}{36} + \frac{15}{36} + \frac{11}{36} + \frac{6}{36} = \frac{7}{2} = 3.5$$

The variance of the random variable, $\bar{x}$:

$$\sigma^2{}_{\bar{x}} = \sum(\bar{x} - \mu)^2 p(\bar{x}) = (1 - 3.5)^2 \frac{1}{36} + (1.5 - 3.5)^2 \frac{1}{18} + (2 - 3.5)^2 \frac{1}{12} + (2.5 - 3.5)^2 \frac{1}{9} + (3 - 3.5)^2 \frac{5}{36} + (3.5 - 3.5)^2 \frac{1}{6} + (4 - 3.5)^2 \frac{5}{36} + (4.5 - 3.5)^2 \frac{1}{9} + (5 - 3.5)^2 \frac{1}{12} + (5.5 - 3.5)^2 \frac{1}{18} + (6 - 3.5)^2 \frac{1}{36} = \frac{6.25}{36} + \frac{8}{36} + \frac{0.75}{36} + \frac{4}{36} + \frac{1.25}{36} + \frac{0}{36} + \frac{1.25}{36} + \frac{4}{36} + \frac{0.75}{36} + \frac{8}{36} + \frac{6.25}{36} = \frac{35}{24}$$

Fig. 2
Sample distribution of the means for two tosses of a six-sided die



Lecturer: Group 4, What is the total number of outcomes when the die is tossed thrice?

Amina: Using the same procedure as we did in the two tosses, we obtained 6 x 6 x 6 = 216 outcomes.

Lecturer: Group 3, list all the possible outcomes and compute the values of the random variables as done in the two tosses.

Akua: Listing each of the outcomes for all 216 outcomes will take a long time to complete. However, we list some of them as (1,1,1), (2,1,1), (3,1,1), (4,1,1), (5,1,1), and so on.

Lecturer: I totally agree, but all groups should make the effort to list all the outcomes. I'm coming round to inspect your results. The complete listing will be 216 outcomes. All the groups are doing well. Tables 4 and 5 indicate sample mean, numerator sum, permutation, frequency and its associated probability.

Table 4
Random variables for three tosses of a die and their sample means

Sample Mean	Numerator Sum	Permutation	Frequency	Sample Mean	Numerator Sum	Permutation	Frequency
3/3	111	3! /3!	1	18/3	666	3! /3!	1

4/3	211	3! /2!	3	17/3	665	3! /2!	3
5/3	311	3! /2!	3	16/3	664	3! /2!	3
	221	3! /2!	3		655	3! /2!	3
Total			6				6
6/3	411	3! /2!	3	15/3	663	3! /2!	3
	312	3!	6		654	3!	6
	222	3! /3!	1		555	3! /3!	1
Total			10				10
7/3	511	3! /2!	3	14/3	662	3! /2!	3
	412	3!	6		653	3!	6
	313	3! /2!	3		644	3! /2!	3
	322	3! /2!	3		554	3! /2!	3
Total			15				15
8/3	611	3! /2!	3	13/3	661	3! /2!	3
	512	3!	6		625	3!	6
	413	3!	6		634	3!	6
	422	3! /2!	3		553	3! /2!	3
	332	3! /2!	3		544	3! /2!	3
Total			21				21
9/3	612	3!	6	12/3	615	3!	6
	513	3!	6		624	3!	6
	522	3! /2!	3		633	3! /2!	3
	423	3!	6		634	3!	6
	441	3! /2!	3		525	3! /2!	3
	333	3! /3!	1		444	3! /3!	1
Total			25				25
10/3	613	3!	6	11/3	614	3!	6
	622	3! /2!	3		623	3!	6
	523	3!	6		524	3!	6
	514	3!	6		515	3! /2!	3
	424	3! /2!	3		533	3! /2!	3
	433	3! /2!	3		434	3! /2!	3
Total			27				27

Table 5
Sample mean, frequency of the sample mean, and its associated probability

$\bar{x}$	Frequency	P ($\bar{x}$)
1	1	1/216
4/3	3	1/72
5/3	6	1/36
2	10	5/108
7/3	15	5/72
8/3	21	7/72
3	25	25/216
10/3	27	1/8
11/3	27	1/8
4	25	25/216
13/3	21	7/72
14/3	15	5/72
5	10	5/108
16/3	6	1/36
17/3	3	1/72
6	1	1/216

Lecturer: All the groups should compute the sample means and variances of the three tosses using the formulae, you used in the two tosses. Group 3, what are your values for the mean of the sample means and the variance?
Osei: We obtained 3.5 and 35/36 for the mean and variance respectively.
Lecturer: What is the relationship between the mean in the three tosses of the die and the mean in one toss?
Osei: The means are the same
Lecturer: Good. Group 3, what is the relationship between the variance in the three tosses of the die and the variance in one toss?
Kofi: By using the previous experience, the variance of the three tosses is equal to the variance of the one toss divided by 3.
Lecturer: Good. All the groups should construct the distribution of the sample means as done previously. All of you have done very well. Group, describe your distribution.
Kojo: Our distribution is bell-shaped and symmetrical about the mean of 3.5.
Lecturer: The distribution is indicated in Fig. 3.

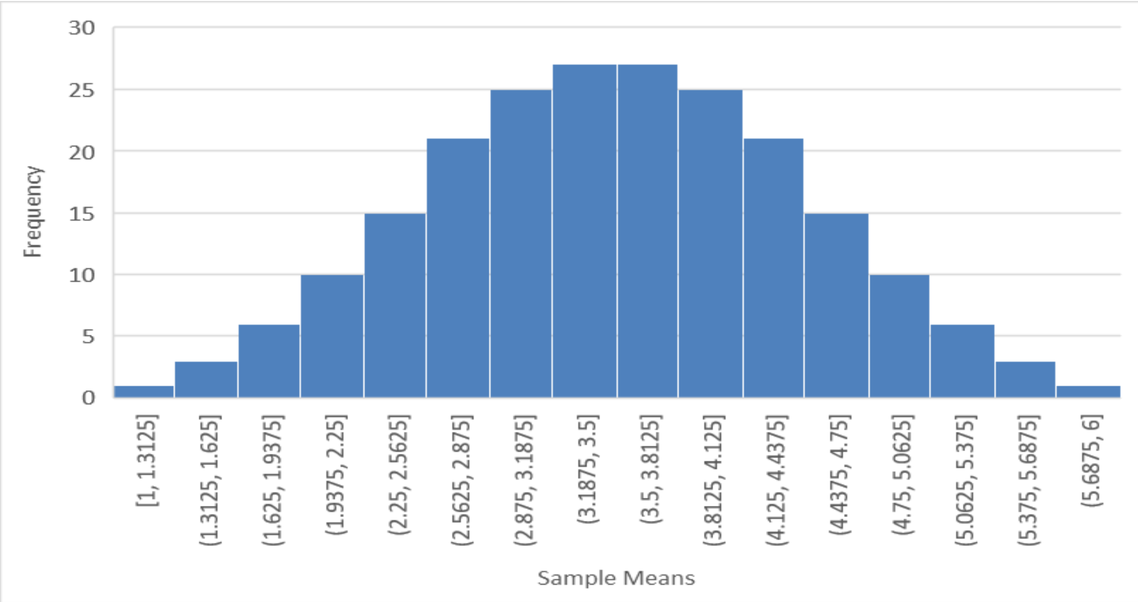
The mean of the random variable $\bar{x}$:

$$\mu_{\bar{x}} = \sum \bar{x}p(\bar{x}) = 1\left(\frac{1}{216}\right) + \frac{4}{3}\left(\frac{1}{72}\right) + \frac{5}{3}\left(\frac{1}{36}\right) + 2\left(\frac{5}{108}\right) + \frac{7}{3}\left(\frac{5}{72}\right) + \frac{8}{3}\left(\frac{7}{72}\right) + 3\left(\frac{25}{216}\right) + \frac{10}{3}\left(\frac{1}{8}\right) + \frac{11}{3}\left(\frac{1}{8}\right) + 4\left(\frac{25}{216}\right) + \frac{13}{3}\left(\frac{7}{72}\right) + \frac{14}{3}\left(\frac{5}{72}\right) + 5\left(\frac{5}{108}\right) + \frac{16}{3}\left(\frac{1}{36}\right) + \frac{17}{3}\left(\frac{1}{72}\right) + 6\left(\frac{1}{216}\right) = \frac{7}{2} = 3.5$$

The variance of the random variable $\bar{x}$:

$$\sigma^2{}_{\bar{x}} = \sum(\bar{x}-\mu)^2 p(\bar{x}) = \left(1-\frac{7}{2}\right)^2\left(\frac{1}{216}\right) + \left(\frac{4}{3}-\frac{7}{2}\right)^2\left(\frac{1}{12}\right) + \left(\frac{5}{3}-\frac{7}{2}\right)^2\left(\frac{1}{36}\right) + \left(2-\frac{7}{2}\right)^2\left(\frac{5}{108}\right) + \left(\frac{7}{3}-\frac{7}{2}\right)^2\left(\frac{5}{72}\right) + \left(\frac{8}{3}-\frac{7}{2}\right)^2\left(\frac{7}{72}\right) + \left(3-\frac{7}{2}\right)^2\left(\frac{25}{216}\right) + \left(\frac{10}{3}-\frac{7}{2}\right)^2\left(\frac{1}{8}\right) + \left(\frac{11}{3}-\frac{7}{2}\right)^2\left(\frac{1}{8}\right) + \left(4-\frac{7}{2}\right)^2\left(\frac{25}{216}\right) + \left(\frac{13}{3}-\frac{7}{2}\right)^2\left(\frac{7}{72}\right) + \left(\frac{14}{3}-\frac{7}{2}\right)^2\left(\frac{5}{72}\right) + \left(5-\frac{7}{2}\right)^2\left(\frac{5}{108}\right) + \left(\frac{16}{3}-\frac{7}{2}\right)^2\left(\frac{1}{36}\right) + \left(\frac{17}{3}-\frac{7}{2}\right)^2\left(\frac{1}{72}\right) + \left(6-\frac{7}{2}\right)^2\left(\frac{1}{216}\right) = \frac{35}{36}$$

Fig. 3
Sample distribution of the means for three tosses of a six-sided die



All the groups should toss the die four times, and record the possible outcomes? Group 2, how many possible outcomes did you get?

Mansa: We got 6 x 6 x 6 x 6 = 6^4=1296 outcomes.
Lecturer: Group 4, go ahead and list all the possible outcomes.
Sampson: Sir, this will take a long time to complete.
Lecturer: Let's devise an easier approach of getting all the outcomes. All groups should note that the possible sample means for the outcomes will be one of these: 1, 5/4, 6/4, 7/4, 8/4, 9/4, 10/4, 11/4, 11/4, 12/4, 13/4, 14/4, 15/4, 16/4, 17/4, 18/4, 19/4, 20/4, 21/4, 22/4, 23/4, 24/4. Table 6 indicate their numerator sum, permutation and frequency Table 7 indicates the sample means; frequencies of the sample means and its associated probability.

Table 6
Sample mean, frequency of the sample mean, and its associated probability

Sample Mean	Numerator Sum	Permutation	Frequency	Sample Mean	Numerator Sum	Permutation	Frequency
4/4	1111	4! /4!	1	24/4	6666	4! /4!	1
5/4	1112	4! /3!	4	23/4	6665	4! /3!	4
6/4	1113	4! /3!	4	22/4	6664	4! /3!	4
	1122	4! /2!2!	6		6655	4! /2!2!	6
Total			10				10
7/4	1114	4! /3!	4	21/4	6663	4! /3!	4
	1123	4! /2!	12		6654	4! /2!	12
	2221	4! /3!	4		6555	4! /3!	4
Total			20				20
8/4	2222	4! /4!	1	20/4	6626	4! /3!	4
	2213	4! /2!	12		6635	4! /2!	12
	2114	4! /2!	12		6644	4! /2!2!	6
	1115	4! /3!	4		6545	4! /2!	12
	3311	4! /2!	6		5555	4! /4!	1
Total			35				35
9/4	2223	4! /3!	4	19/4	6616	4! /3!	4
	2214	4! /2!	12		6625	4! /2!	12
	2115	4! /2!	12		6634	4! /2!	12
	2133	4! /2!	12		6544	4! /2!	12
	6111	4! /3!	4		6535	4! /2!	12
	1143	4! /2!	12		5545	4! /3!	4
Total			56				56
10/4	3331	4! /3!	4	18/4	6615	4! /2!	12
	3322	4! /2!2!	6		6624	4! /2!	12
	4222	4! /3!	4		6633	4! /2!2!	6
	4321	4!	24		6534	4!	24
	4411	4! /2!2!	6		6525	4! /2!	12
	5311	4! /2!	12		6444	4! /3!	4

	6211	4! /2!	12		5544	4! /2!2!	6
	5221	4! /2!	12		5535	4! /3!	4
Total			80				80
11/4	6611	4! /2!	12	17/4	6614	4! /2!	12
	6221	4! /2!	12		6623	4! /2!	12
	5411	4! /2!	12		6515	4! /2!	12
	5222	4! /3!	4		6524	4!	24
	5132	4!	24		6533	4! /2!	12
	4412	4! /2!	12		6434	4! /2!	12
	4322	4! /2!	12		5525	4! /3!	4
	3341	4! /2!	12		5534	4! /2!	12
	3323	4! /3!	4		4445	4! /3!	4
Total			104				104
12/4	6411	4! /2!	12	16/4	6613	4! /2!	12
	6321	4!	24		6622	4! /2!2!	6
	6222	4! /3!	4		6514	4!	24
	5511	4! /2!2!	6		6523	4!	24
	5412	4!	24		6424	4! /2!	12
	5322	4! /2!	12		6433	4! /2!	12
	5133	4! /2!	12		5515	4! /3!	4
	4422	4! /2!2!	6		5524	4! /2!	12
	4413	4! /2!	12		5533	4! /2!2!	6
	3342	4! /2!	12		4444	4! /4!	1
	3333	4! /4!	12		4435	4! /2!	12
Total			125				125
13/4	6511	4! /2!	12	15/4	6612	4! /2!	12
	6412	4!	24		6513	4!	24
	6322	4! /2!	12		6522	4! /2!	12
	6313	4! /2!	12		6423	4!	24
	5512	4! /2!	12		6414	4! /2!	12
	5422	4! /2!	12		6333	4! /3!	4
	5413	4!	24		5514	4! /2!	12
	4423	4! /2!	12		5523	4! /2!	12
	4414	4! /3!	4		4434	4! /3!	4
	4333	4! /3!	4		4425	4! /2!	12
	3325	4! /2!	12		3345	4! /2!	12
Total			140				140
14/4	6611	4! /2!2!	6				
	6512	4!	24				
	6413	4!	24				
	6422	4! /2!	12				
	5513	4! /2!	12				
	5522	4! /2!2!	6				
	4415	4! /2!	12				
	4424	4! /2!	12				
	4433	4! /2!2!	6				
	3335	4! /3!	4				
	3326	4! /2!	12				
	4532	4!	24				
Total			146				

Table 7
Sample mean, frequency of the sample mean, and its associated probability

$\bar{x}$	Frequency	$p(\bar{x})$
1	1	1/1296
5/4	4	1/324
6/4	10	5/648
7/4	20	5/324
8/4	35	35/1296
9/4	56	7/162
10/4	80	5/81
11/4	104	13/162
12/4	125	125/1296
13/4	140	35/324
14/4	146	73/648

15/4	140	35/324
16/4	125	125/1296
17/4	104	13/162
18/4	80	5/81
19/4	56	7/162
20/4	35	35/1296
21/4	20	5/324
22/4	10	5/648
23/4	4	1/324
6	1	1/1296

Lecturer: All the groups should compute the sample means and variances of the three tosses using the formulae you used in the two tosses. Group 3, what are your values for the mean of the sample means and the variance?
Osei: We obtained 3.5 and 35/36 for the mathematical expectation and variance respectively.
Lecturer: What is the relationship between the mean in the three tosses of the die and the mean in one toss?
Osei: The means are the same.
Lecturer: Good. Group 3, what is the relationship between the variance in the three tosses of the die and the variance in one toss?
Kofi: By using the previous experience, the variance of the three tosses is equal to the Variance in one toss divided by 3.
Lecturer: Good. All the groups should construct the distribution of the sample means as done previously.
Lecturer: All of you have done very well. Group 4, describe your distribution.
Kojo: Our distribution is bell-shaped, its symmetrical about the mean of 3.5.
Lecturer: The distribution is indicated in Fig. 4.

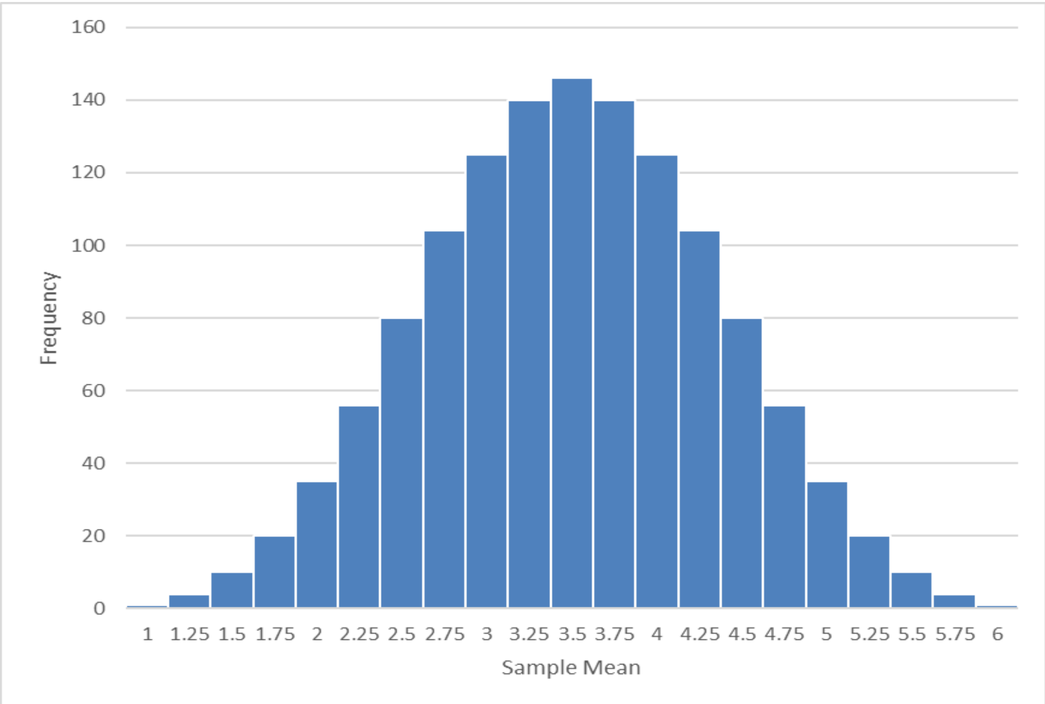
The mean of the random variable $\bar{x}$

$$\mu_x = \sum \bar{x}p(\bar{x}) = 1\left(\frac{1}{1296}\right)+\frac{5}{4}\left(\frac{1}{324}\right)+\frac{6}{4}\left(\frac{5}{648}\right)+\frac{7}{4}\left(\frac{5}{324}\right)+\frac{8}{4}\left(\frac{35}{1296}\right)+\frac{9}{4}\left(\frac{7}{162}\right)+\frac{10}{4}\left(\frac{5}{81}\right)+\frac{11}{4}\left(\frac{13}{162}\right)+\frac{12}{4}\left(\frac{125}{1296}\right)+\frac{13}{4}\left(\frac{35}{1296}\right)+\frac{14}{4}\left(\frac{73}{648}\right)+\frac{15}{4}\left(\frac{35}{324}\right)+\frac{16}{4}\left(\frac{125}{1296}\right)+\frac{17}{4}\left(\frac{13}{162}\right)+\frac{18}{4}\left(\frac{5}{81}\right)+\frac{19}{4}\left(\frac{7}{162}\right)+\frac{20}{4}\left(\frac{35}{1296}\right)+\frac{21}{4}\left(\frac{5}{324}\right)+\frac{22}{4}\left(\frac{5}{648}\right)+\frac{23}{4}\left(\frac{1}{324}\right)+\frac{24}{4}\left(\frac{1}{1296}\right)=\frac{7}{2}=3.5$$

The variance of the randon variable, $\bar{x}$:

$$\sigma^2{}_{\bar{x}} = \sum(\bar{x}-\mu)^2 p(\bar{x}) = \left(1-\frac{7}{2}\right)^2\left(\frac{1}{1296}\right)+\left(\frac{5}{4}-\frac{7}{2}\right)^2\left(\frac{1}{324}\right)+\left(\frac{6}{4}-\frac{7}{2}\right)^2\left(\frac{5}{648}\right)+\left(\frac{7}{4}-\frac{7}{2}\right)^2\left(\frac{5}{324}\right)+\left(\frac{8}{4}-\frac{7}{2}\right)^2\left(\frac{35}{1296}\right)+\left(\frac{9}{4}-\frac{7}{2}\right)^2\left(\frac{7}{162}\right)+\left(\frac{10}{4}-\frac{7}{2}\right)^2\left(\frac{5}{81}\right)+\left(\frac{11}{4}-\frac{7}{2}\right)^2\left(\frac{13}{162}\right)+\left(\frac{12}{4}-\frac{7}{2}\right)^2\left(\frac{125}{1296}\right)+\left(\frac{13}{4}-\frac{7}{2}\right)^2\left(\frac{35}{1296}\right)+\left(\frac{14}{4}-\frac{7}{2}\right)^2\left(\frac{73}{648}\right)+\left(\frac{15}{4}-\frac{7}{2}\right)^2\left(\frac{35}{324}\right)+\left(\frac{16}{4}-\frac{7}{2}\right)^2\left(\frac{125}{1296}\right)+\left(\frac{17}{4}-\frac{7}{2}\right)^2\left(\frac{13}{162}\right)+\left(\frac{18}{4}-\frac{7}{2}\right)^2\left(\frac{5}{81}\right)+\left(\frac{19}{4}-\frac{7}{2}\right)^2\left(\frac{7}{162}\right)+\left(\frac{20}{4}-\frac{7}{2}\right)^2\left(\frac{35}{1296}\right)+\left(\frac{21}{4}-\frac{7}{2}\right)^2\left(\frac{5}{324}\right)+\left(\frac{22}{4}-\frac{7}{2}\right)^2\left(\frac{5}{648}\right)+\left(\frac{23}{4}-\frac{7}{2}\right)^2\left(\frac{1}{324}\right)+\left(\frac{24}{4}-\frac{7}{2}\right)^2\left(\frac{1}{1296}\right)=\frac{35}{48}.$$

Fig. 4
Sample distribution of the means for four tosses of a six-sided die



Lecturer: Groups, we will all move to the computer laboratory and complete the next task individually. Everyone of you should get himself or herself seated by a computer. Use Microsoft Excel to generate three random numbers between 1 and 6, each column indicating the toss of a die. Therefore, the three columns indicate three tosses of the die. Compute the mean of the three samples. Repeat the process for 10,000 simulations, and draw a graph of the distribution of the sample means.
Lecturer: Amina, What is the shape of the distribution?
Bell-shaped, but looking more of a normal curve than in the two previous tosses of the die.
Please, repeat the process for 6 random numbers between 1 and 6, each column indicating the toss of the die, and compute the mean of the six samples. Repeat the process for 10,000 simulations and draw the graph of the distribution of the sample means.
Lecturer: What do you notice about the two distributions?

They all look almost bell-shaped, but the number of bins in the latter is more than in the former. Aside that, the bin lengths decrease with increase in samples and also the second distribution looks more normal than the first.

We've consdered six experiments to help you understand the sampling distribution of the sample means. Can anyone of you tell the whole class his/her observations in all of these experiments?

Musa: I've observed that the larger the sample size, the more normal the sample distribution of the sample means.

Akosua: That's true, particularly in the two simulations, the second with a sample size of six produed a distribution, more normal than the former.

Lecturer: The distributions are indicated in Figures 7 and 8.

Fig. 7

Sample distribution of the means for three randomly generated numbers in 10,000 simulations

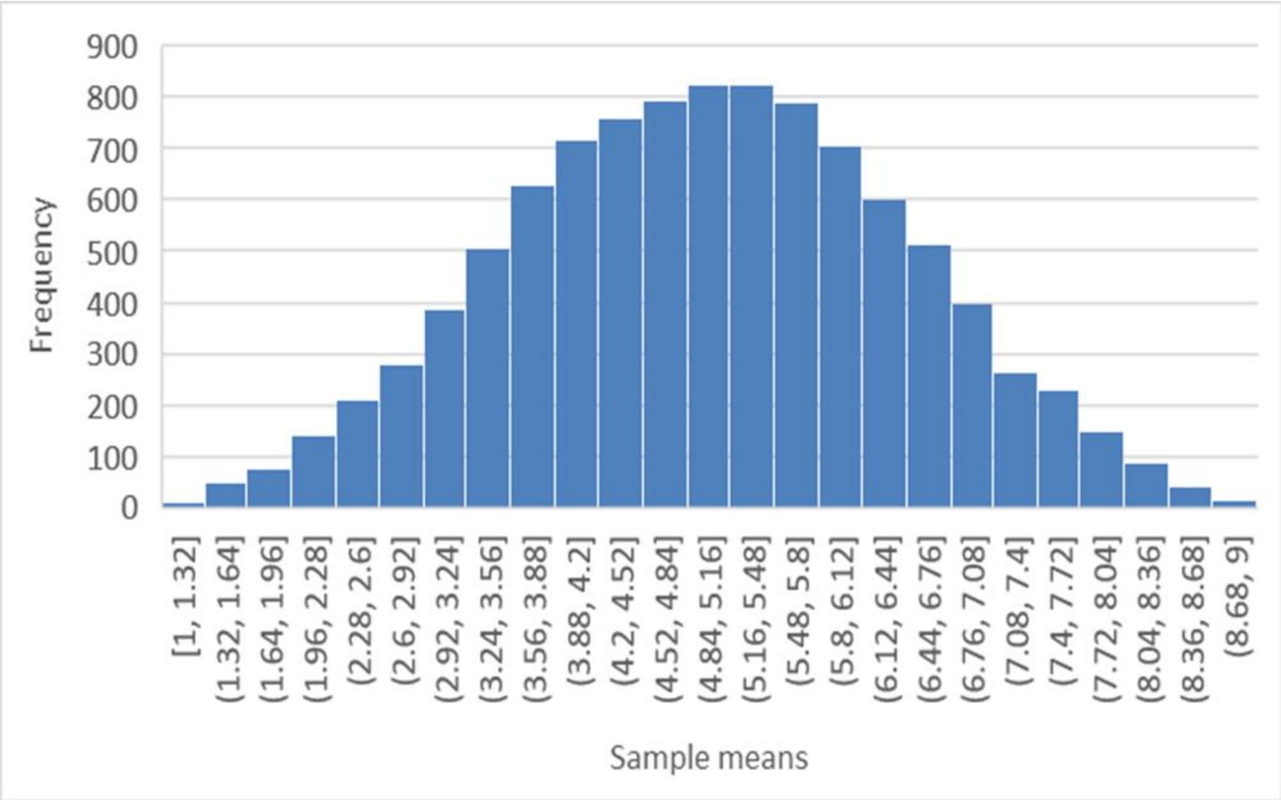
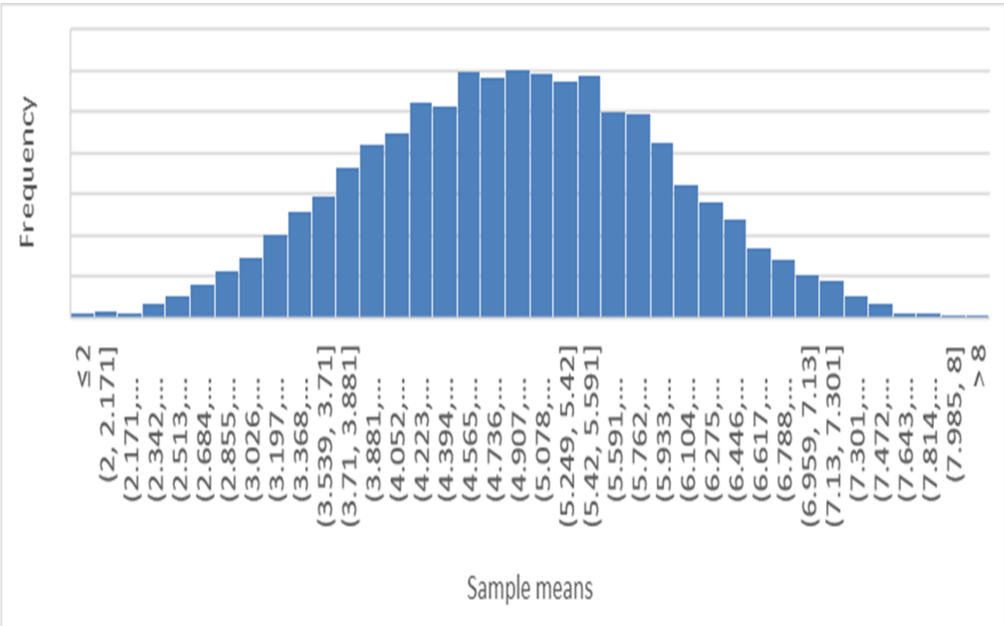


Fig. 8

Sample distribution of the means for six randomly generated numbers in 10,000 simulations



Good to hear that. We've arrived at an important theorem in statistics called the Central Limit Theorem. It states that:If random samples of size n are selected from a population with mean μ and variance σ^2, the sampling distribution of the sample mean, $\bar{x}$, will be approximately normally distributed, with mean $\mu_{\bar{x}} = \mu$ and variance ${\sigma^2}_{\bar{x}} = \frac{\sigma^2}{n}$.

Alternatively, If $X_1, X_2, \ldots, X_n$ are i.i.d. (independent and identically distributed) random variables having the same distribution with mean μ, and variance σ^2, and moment generating function $M_X(t)$, then if $n \to \infty$, the limiting distribution of the random variable $Z = \frac{T-n\mu}{\sigma\sqrt{n}}$ (where $T = T_1 + T_2 + \cdots + T_n$) is the standard normal distribution N(0,1).

Proof:

$$M_z(t) = M_{\frac{T-n\mu}{\sigma\sqrt{n}}}(t) = Ee^{\frac{T-n\mu}{\sigma\sqrt{n}}t} = e^{-\frac{n\mu}{\sigma\sqrt{n}}} M_T\left(\frac{t}{\sigma\sqrt{n}}\right)$$

But $T = X_1 + X_2 + \cdots + X_n$. From earlier discussion the moment generating function of he sum is equal to the product of the individual moment generating function. Hear, each X_i has moment generating function $M_X(t)$. Therefore, $M_T\left(\frac{t}{\sigma\sqrt{n}}\right) = \left[M_X \frac{t}{\sigma\sqrt{n}}\right]^n$ and so $M_Z(t) = e^{-\frac{n\mu}{\sigma\sqrt{n}}t}\left[M_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n$. One way to find the limit of $M_Z(t)$ as $n \to \infty$ is to consider the logaritm of $M_Z(t)$:

$lnM_Z(t) = -\frac{\sqrt{n}\mu}{\sigma}t + n\, lnM_X\left(\frac{t}{\sigma\sqrt{n}}\right)$. Expanding $M_X\left(\frac{t}{\sigma\sqrt{n}}\right)$ gives

$$M_X(t) = \sum_x P(x) + \frac{t}{1!}\sum_x xP(x) + \frac{t^2}{2!}\sum_x x^2P(x) + \frac{t^3}{3!}\sum_x x^3\,P(x) + \cdots$$

We get

$$\ln M_Z(t) = -\frac{\sqrt{n}\mu}{\sigma}t + n4\left(ln\left[1 + \frac{\frac{t}{\sigma\sqrt{n}}}{1!}EX + \frac{\left(\frac{t}{\sigma\sqrt{n}}\right)^2}{2!}EX^2 + \frac{\left(\frac{t}{\sigma\sqrt{n}}\right)^3}{3!}EX^3 + \cdots\right]\right)$$

Using the series expansion of $ln(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots$ where

$y = \frac{\frac{t}{\sigma\sqrt{n}}}{1!}EX + \frac{\left(\frac{t}{\sigma\sqrt{n}}\right)^2}{2!}EX^2 + \frac{\left(\frac{t}{\sigma\sqrt{n}}\right)^3}{3!}EX^3 + \cdots$ we get:

$$\ln M_Z(t) = -\frac{\sqrt{n}\mu}{\sigma} + n\left[\frac{\frac{t}{\sigma\sqrt{n}}}{1!}EX + \frac{\left(\frac{t}{\sigma\sqrt{n}}\right)^2}{2!}EX^2 + \frac{\left(\frac{t}{\sigma\sqrt{2}}\right)^2}{3!}EX^3 + \cdots\right]$$

$$-\frac{1}{2}\left[\frac{\frac{t}{\sigma\sqrt{n}}}{1!}EX + \frac{\left(\frac{t}{\sigma\sqrt{n}}\right)^2}{2!}EX^2 + \frac{\left(\frac{t}{\sigma\sqrt{n}}\right)^3}{3!}EX^3 + \cdots\right]^2$$

$$+\frac{1}{3}\left[\frac{\frac{t}{\sigma\sqrt{n}}}{1!}EX + \frac{\left(\frac{t}{\sigma\sqrt{n}}\right)^2}{2!}EX^2 + \frac{\left(\frac{t}{\sigma\sqrt{n}}\right)^3}{3!}EX^3 + \cdots\right]^3$$

Factoring out the powers of t we obtain:

$$\ln M_Z(t) = \left(-\frac{\sqrt{n}\mu}{\sigma} + \frac{\sqrt{n}EX}{\sigma}\right)t + \left(\frac{EX^2}{2\sigma^2} - \frac{(EX)^2}{2\sigma^2}\right)t^2$$

$$+\left(\frac{EX^3}{6\sigma^3\sqrt{N}} - \frac{EX\,EX^2}{2\sigma^3\sqrt{n}} + \frac{(EX)^3}{3\sigma^3\sqrt{n}}\right)t^3 + \cdots$$

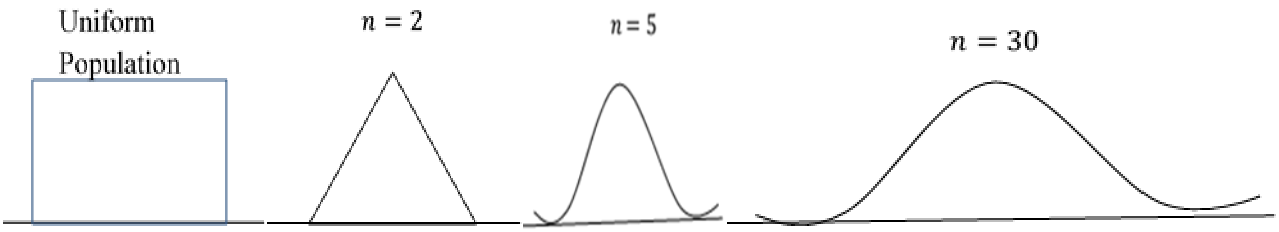
Because $EX = \mu$ and $EX^2 - (EX)^2 = \sigma^2$, the last expression becomes

$$\ln M_Z(t) = \frac{1}{2}t^2 + \left(\frac{EX^3}{6} - \frac{EX\,EX^3}{2} + \frac{(EX)^3}{3}\right)\frac{t^3}{\sigma^3\sqrt{n}} + \cdots$$

We observe that as $n \to \infty$ the limit of the previous expression is $\lim_{n\to\infty} \ln M_Z(t) = \frac{1}{2}t^2$, so $\lim_{n\to\infty} M_Z(t) = e^{\frac{t^2}{2}}$. But this is the moment generating function of the standard normal distribution. Therefore the limiting distribution of $\frac{T-n\mu}{\sigma\sqrt{n}}$ is the standard normal distribution $N(0,1)$. All of you should note that the shape of the population and the sample size are very important to successfully apply the Central Limit Theorem. The possible conditions are as follows: If the sample size is large ($n \geq 30$), the sampling distribution of $\bar{x}$ is approximately normal, regardless of the shape of the population. If the sample size is small ($n < 30$), the sampling distribution of $\bar{x}$ is approximately normal, provided that the shape of the population is not drastically different from normal. The Central Limit Theorem does not apply if the sample size is small ($n < 30$) and the shape the population resembles (for example) an inverted normal curve, which is generally referred to as U-shaped. Figures 5 and 6 show the sampling distributions of the samples means when sample sizes of 2, 5, and 30, are drawn from the uniform, exponential, U-shaped and normal populations. In each case, the sampling distribution is normal when n = 30, irrespective of the shape of the population. It needs emphasizing that the Central limit Theorem provides a basis for testing hypothesis, and making inferences.

Fig. 5

Sampling distributions of uniform and exponential, populations when n =2, 5, and 30



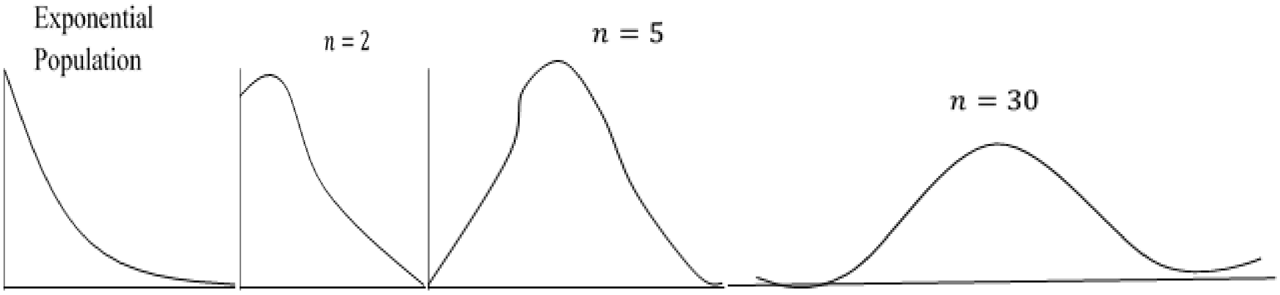
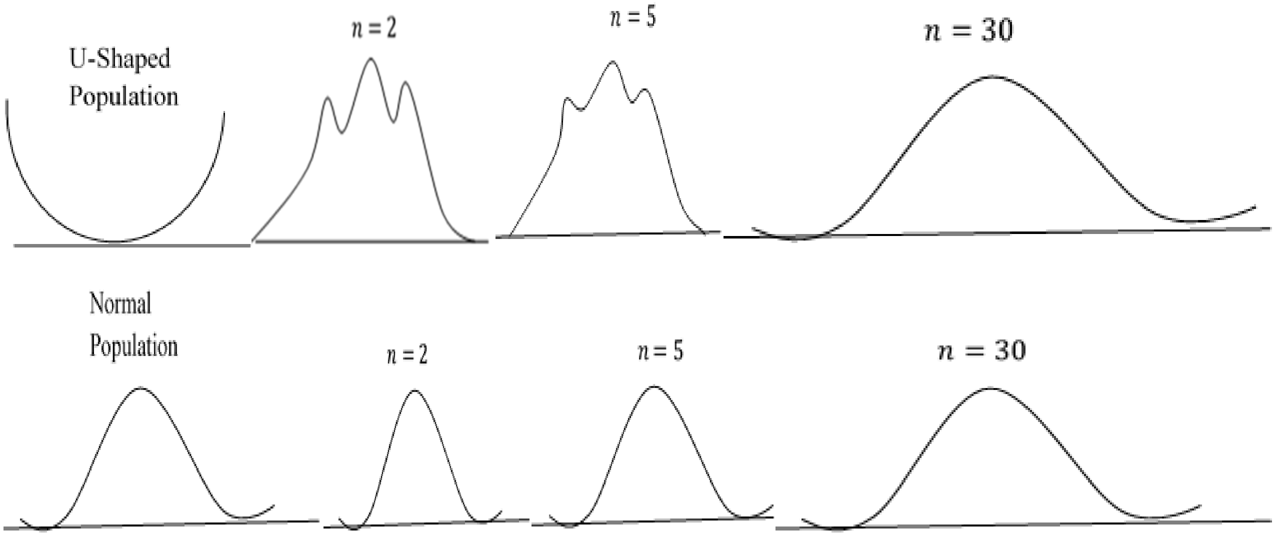


Fig.6
Sampling distributions of U-shaped and normal populations when n =2, 5, 30



DISCUSSION:
Under the guidance of their lecturers, the students explored mathematical facts and obtained mathematical evidence to support their claims (Crede, 2008). The students understood and applied statistical concepts, because their lecturer had introduced them to simple and easy-to-understand concepts, before introducing them to complex ones. For instance, understanding the concepts (Rubin, et al., 1990; Shaughnessy, 1992; Tversky & Kahneman, 1971), such as mathematical expectation (mean), variance, and probability, as well as, their computations, were critical in understanding the sampling distribution of the means. Students would be able to understand the concepts and apply them in multiple contexts if this becomes lecturers' professional practice (Kuncel, 2008). It is worth emphasizing that students' ability to understand the concept of the sampling distribution of the mean in the first few tosses of the die, was so critical in enhancing their subsequent understanding in higher tosses.

The students' procedural knowledge in statistics is greatly improved as a result. This knowledge is "knowing how" or the knowledge of the steps required to attain various goals" (Bynes & Wasik, 1991, p.777). Through guided- inquiry, the students calculated the mathematical expectation (mean) and variance, of a sample. Listing all the sample spaces, especially, in the higher tosses of the die, was made easy by using permutation, a counting procedure involving order restrictions. In all these, the students gained greater autonomy in the learning process (Crede, 2008). The students thought critically and expressed themselves clearly and confidently when solving problems (Bransford, et al., 2000).

The use of simulations provided the students with a real experience that required their complete active involvement and participation (Hertel & Millis, 2002). This allowed them to develop leadership skills and become more adept at analysing and solving issues. The simulations promoted a transfer of knowledge and helped with the application of the concepts learnt. Ultimately, the students learnt how to think critically in a complex situation (Brumfield, 2005).

Through simulations, the students had an opportunity to participate in active learning and make constructive decisions. Through these group exercises, they gained a better understanding of group dynamics and processes. Simulations allowed for a deeper exploration of complex issues or concepts with greater student involvement in the learning experience (Coffman, 2006), and made the students aware of their thought processes and how they arrived at certain conclusions. The students built social and emotional learning skills, that enhanced their problem-solving skills. The students cultivated skills in their group discussions and applied reasoning to reach conclusions.

Simulations provided the students with realistic experience, which they transferred to new problems and situations as they fidgeted with their computers to get the expected result (Coffman, 2006). They enhanced the students' learning and increased their interest and awareness in the sampling distribution of the sample means. They provided opportunities for them to explore situations that mirrored real-world ones. They provided the students with an opportunity to practice problem-based learning using the tasks assigned to them. These tasks enabled them to work together to find a possible solution. It is important to note that there was no one quick solution (Gredler, 1992).

The lecturer assigned the students roles and expected them to act within those roles or setting for the tasks. They carried out their roles according to predetermined behavioural characteristics and descriptions regarding the tasks. The students took ownership and responsibility for their learning. The lecturer, through the process, functioned only as a facilitator (Hertel & Millis, 2002). The students gained an understanding as they interacted with their peers, through social negotiation (Kirkley & Kirkley, 2005).

Conclusion

Using the guided inquiry teaching method to explain statistical concepts to students has proven to be very effective. As students perform specific tasks, either individually or in groups, they gain autonomy of their learning and internalize the concept-building processes. They can link previous concepts to current ones, and

master procedurally the steps required to solve problems accurately. The use of simulations in this study has also proven to be germane in painting a real picture of the complex scenarios, which hitherto, would have been very difficult for students to comprehend.

IMPLICATION FOR TEACHING AND LEARNING:

As much as practicable, teachers should use instructional strategies that allow students to construct their mathematical understanding. Beginning from simple scenarios and transitioning into complex ones, they should allow students to compute statistical values and interpret them. Teachers must note that developing students' conceptual knowledge is paramount, if we want our students, in their forward march, to become effective problem solvers.

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